

Rotation number of a unimodular cycle: an elementary approach

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Abstract

We give an elementary proof of a formula expressing the rotation number of a cyclic unimodular sequence $L = u_1 u_2 \dots u_d$ of lattice vectors $u_i \in \mathbb{Z}^2$ in terms of arithmetically defined local quantities. The formula has been originally derived by Akihiro Higashitani and Mikiya Masuda (arXiv:1204.0088v2 [math.CO]) with the aid of the Riemann-Roch formula applied in the context of toric topology. They also demonstrated that a generalized versions of the “Twelve-point theorem” and a generalized Pick’s formula are among the consequences of their result so our proof opens a way to an elementary approach to these results as well.

Theorem 1. *The rotation number $Rot(L)$ of a cyclic unimodular sequence $L = v_1 v_2 \dots v_d$ can be calculated as the weighted sum*

$$Rot(L) = \frac{1}{12}\mu(L) + \frac{1}{4}\nu(L) = \frac{1}{12} \sum_{i=1}^d \mu(v_{i-1}, v_i, v_{i+1}) + \frac{1}{4} \sum_{i=1}^d \nu(v_i, v_{i+1}) \quad (1)$$

of locally defined quantities $\mu(L)$ and $\nu(L)$ where $\nu(v_i, v_{i+1}) := \det(v_i, v_{i+1}) \in \{-1, +1\}$ and $\mu(v_{i-1}, v_i, v_{i+1}) = a_i \in \mathbb{Z}$ is the integer determined by the equation

$$\det(v_{i-1}, v_i)v_{i-1} + \det(v_i, v_{i+1})v_{i+1} + a_i v_i = 0.$$

1 Introductory remarks and definitions

The formula (1) exhibits interesting similarities and differences when compared with more conventional formulas for the rotation number of a planar curve Γ , say the formula

$$Rot(\Gamma) = \frac{1}{2\pi} \int_{\Gamma} \frac{xdy - ydx}{x^2 + y^2}. \quad (2)$$

Both formulas are local, expressing the rotation number as a sum (integral) of quantities defined locally on the curve. Both formulas apply to closed curves, considering that each cyclic unimodular sequence L defines a closed unimodular polygonal line P_L (Section 1.1). On closer inspection they even display an interesting analogy between the set S^1 of unit vectors in \mathbb{R}^2 and the set $Prim(\mathbb{Z}^2)$ of primitive lattice vectors. Indeed, the local quantity integrated in (2) is the (infinitesimal) angle $d\theta = d \arctan(y/x)$ where $\theta : \mathbb{R}^2 \setminus \{0\} \rightarrow S^1$ is the function $\theta(v) = \|v\|^{-1}v$, whereas a similar role of “discrete angle functions” is in formula (1) played by the functions μ and ν .

1.1 Unimodular sequences and unimodular cycles

A sequence $L = u_1 u_2 \dots u_d$ of lattice vectors is called *unimodular* if $\{u_i, u_{i+1}\}$ is a base of the lattice \mathbb{Z}^2 for each i or equivalently if $\det(u_i, u_{i+1}) \in \{-1, +1\}$ for each $i = 1, \dots, d-1$.

A unimodular sequence $L = u_1 u_2 \dots u_d$ is called *cyclic* if $\det(u_d, u_1) \in \{-1, +1\}$. A cyclic unimodular sequence L naturally defines a periodic unimodular sequence $W = \dots LLL \dots = \dots u_{-1} u_0 u_1 u_2 \dots u_d u_{d+1} \dots$ where $u_i = u_j$ if $i \equiv j \pmod{d}$. For this reason it may be convenient to use occasionally the notation $L = u_1 u_2 \dots u_d(u_1)$ which emphasizes that L is a circular, unimodular sequence of length d .

Suppose that $L = u_1 u_2 \dots u_d u_{d+1}$ is a unimodular sequence which is *antipodal* in the sense that $u_{d+1} = -u_1$. Then its *centrally symmetric extension* L^c is the cyclic, unimodular sequence $L' := u_1 \dots u_d(-u_1) \dots (-u_d)$ of length (period) $2d$.

Given a cyclic, unimodular sequence $L = u_1 u_2 \dots u_d(u_1)$ the associated *closed unimodular polygon* P_L is the closed, oriented polygonal curve which

has u_i as vertices and $[u_i, u_{i+1}]$, $i = 1, \dots, d$ as line segments. The fundamental class of P_L is the 1-chain

$$[L] := \overrightarrow{u_1 u_2} + \overrightarrow{u_2 u_3} + \dots + \overrightarrow{u_{d-1} u_d} + \overrightarrow{u_d u_1}$$

which is referred to as the *unimodular cycle* associated to L .

The rotation or the winding number $Rot(L)$ of a cyclic, unimodular sequence $L = u_1 u_2 \dots u_d$ is defined as the rotation number of the (oriented, closed) polygonal curve P_L or equivalently as the homology class of the unimodular cycle $[L]$ in $H_1(\mathbb{R}^2 \setminus \{0\})$.

The invariants $\mu(L)$ and $\nu(L)$ of a (not necessarily cyclic) unimodular sequence $L = u_1 \dots u_d$ are already introduced in Theorem 1 as the sums

$$\mu(L) = \sum_{i=1}^d \mu(u_{i-1}, u_i, u_{i+1}) \quad \nu(L) = \sum_{i=1}^d \nu(u_i, u_{i+1}) \quad (3)$$

where $\nu(u_i, u_{i+1}) := \det(u_i, u_{i+1}) \in \{-1, +1\}$ and $\mu(u_{i-1}, u_i, u_{i+1}) = a_i \in \mathbb{Z}$ is the integer determined by the equation

$$\det(u_{i-1}, u_i)u_{i-1} + \det(u_i, u_{i+1})u_{i+1} + a_i u_i = 0. \quad (4)$$

2 Invariant μ

Lemma 2. *Suppose that $u, v, w \in \mathbb{R}^2$. Then,*

$$\det(u, v)w + \det(v, w)u + \det(w, u)v = 0 \quad (5)$$

Moreover, if $\text{Span}(u, v, w) = \mathbb{R}^2$ then (5) is up to a scalar factor the only dependence between vectors u, v, w .

Proposition 3. *Suppose that (w, u, v) is an ordered triple of lattice vectors such that (u, v) and (v, w) are ordered bases of \mathbb{Z}^2 . Then,*

$$\mu(w, u, v) = \det(w, u)\det(u, v)\det(v, w). \quad (6)$$

Proof: By definition $\mu(w, u, v) = a$ is the integer determined by the equation

$$\det(w, u)w + \det(u, v)v + au = 0.$$

A multiplication of both sides of this equality by $\det(w, u)\det(u, v)$ and a comparison with (5) yields the desired formula. \square

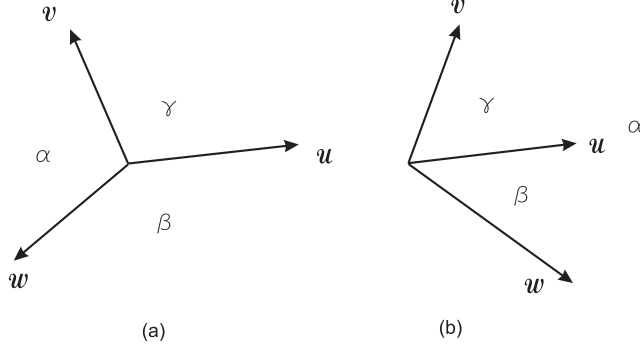


Figure 1: Cyclic unimodular sequences of length 3.

Corollary 4. *If u, v, w is a unimodular sequence then $\mu(w, v, u) = -\mu(u, v, w)$ and $\mu(-u, -v, -w) = \mu(u, v, w)$. More generally $\mu(O(u), O(v), O(w)) = \det(O)\mu(u, v, w)$ for each $O \in GL(\mathbb{Z}, 2)$.*

Corollary 5. *Suppose a, x, b is a unimodular sequence. Then both $a \pm x, x, b$ and $a, x, b \pm x$ are unimodular and,*

$$\begin{aligned}\mu(a \pm x, x, b) &= \mu(a, x, b) \pm \det(x, b) \\ \mu(a, x, b \pm x) &= \mu(a, x, b) \pm \det(a, x)\end{aligned}$$

Corollary 6. *Assume that $L = (u, v, w)$ is a cyclic unimodular sequence. Let $\alpha := \det(v, w)$, $\beta := \det(w, u)$ and $\gamma := \det(u, v)$ (Figure 1). Then*

$$\mu(L) = 3\alpha\beta\gamma \quad \text{and} \quad \nu(L) = \alpha + \beta + \gamma.$$

Example 7. As an illustration let us check Theorem 1 for the case of a cyclic unimodular sequence $L = (u, v, w)$. Assume that $0 \in \text{conv}\{u, v, w\}$ and that the sequence is positively oriented (Figure 1 (a)) which means that $\alpha = \beta = \gamma = 1$. Then $\text{Rot}(u, v, w) = 1$ while $\mu(L) = \nu(L) = 3$. Similarly if $-\alpha = \beta = \gamma = 1$ (Figure 1 (b)) then $\text{Rot}(L) = 0$, $\mu(L) = -3$ and $\nu(L) = 1$. In both cases $\text{Rot}(L) = (1/12)\mu(L) + (1/4)\nu(L)$.

2.1 Exchange lemmas and the Jacobi identity

Proposition 8. (First Exchange Lemma) *Suppose that a, x, b and u, x, v are (not necessarily cyclic) unimodular sequences. Then the sequences a, x, v and u, x, b are also unimodular and*

$$\mu(a, x, b) + \mu(u, x, v) = \mu(a, x, v) + \mu(u, x, b). \quad (7)$$

Proof: Let us write the defining equations of the corresponding μ -invariants.

$$\begin{aligned}\det(a, x)a + \det(x, b)b + \mu(a, x, b)x &= 0 \\ \det(u, x)u + \det(x, v)v + \mu(u, x, v)x &= 0 \\ \det(a, x)a + \det(x, v)v + \mu(a, x, v)x &= 0 \\ \det(u, x)u + \det(x, b)b + \mu(u, x, b)x &= 0\end{aligned}$$

The equality (7) is obtained by subtracting the third and fourth equation from the sum of the first two equations. \square

Proposition 9. (Second Exchange Lemma) *Suppose that a, x, b and u, y, v are unimodular sequences such that $x + y = 0$. Then $a, x, -v$ and $u, y, -b$ are also unimodular and*

$$\mu(a, x, b) + \mu(u, y, v) = \mu(a, x, -v) + \mu(u, y, -b). \quad (8)$$

Proof: In light of the relation $y = -x$, the identity follows from the comparison of the defining equations for the corresponding μ -invariants.

$$\begin{aligned}\det(a, x)a + \det(x, b)b + \mu(a, x, b)x &= 0 \\ \det(u, y)u + \det(y, v)v + \mu(u, y, v)y &= 0 \\ \det(a, x)a + \det(x, -v)(-v) + \mu(a, x, -v)x &= 0 \\ \det(u, y)u + \det(y, -b)(-b) + \mu(u, y, -b)y &= 0\end{aligned}$$

Proposition 10. (Jacobi identity) *Suppose that $(a, x), (b, x), (c, x)$ are bases of the lattice \mathbb{Z}^2 which implies the unimodularity of $C = (a, x, b), A = (b, x, c)$ and $B = (c, x, a)$. Then,*

$$\mu(a, x, b) + \mu(b, x, c) + \mu(c, x, a) = 0 \quad (9)$$

$$\mu(a, x, b) = \mu(a, x, c) + \mu(c, x, b) \quad (10)$$

Proof: Since $\mu(z, y, x) = -\mu(x, y, z)$ (Corollary 4) the equalities (9) and (10) are equivalent so it is sufficient to prove only one of them.

The Plücker relation associated to the 2×4 matrix $M = [a \ b \ c \ x]$ is

$$\det(a, b)\det(c, x) - \det(a, c)\det(b, x) + \det(a, x)\det(b, c) = 0.$$

On multiplying both sides of this equality by $\det(a, x)\det(b, x)\det(c, x)$ we obtain the relation

$$\det(a, b)\det(a, x)\det(b, x) - \det(a, c)\det(a, x)\det(c, x) + \det(b, c)\det(b, x)\det(c, x)$$

which is by Proposition 3 equivalent to (9). \square

3 The additivity of μ and ν

In this section we describe a reduction procedure for cyclic unimodular sequences which serves as a basis for an inductive proof of the formula (1). The general idea is to express a given sequence L (or rather the associated cycle $[L]$) as a sum $L = L_1 + L_2$ of simpler sequences such that $\mu(L) = \mu(L_1) + \mu(L_2)$ and $\nu(L_1) + \nu(L_2)$. Since the rotation number has a similar behavior $\text{Rot}(L) = \text{Rot}(L_1) + \text{Rot}(L_2)$, this eventually reduces (1) to the case of cyclic (and antipodal, Section 3.4) unimodular sequences of length 3 where the formula is easily verified (Examples 7 and 16).

3.1 Removing self-intersections

Suppose that $L = \dots u_{-1}xu_{+1} \dots v_{-1}xv_{+1} \dots$ is a cyclic unimodular sequence which has a *self-intersection* in the sense that a primitive vector x appears twice in L , Figure 2 (a). Then L allows a “shortcut decomposition” $L = L_1 + L_2$ (more precisely $[L] = [L_1] + [L_2]$) where $L_1 = \dots u_{-1}xv_{+1} \dots$ and $L_2 = \dots v_{-1}xu_{+1} \dots$ are subsequences of L , Figure 2 (a).

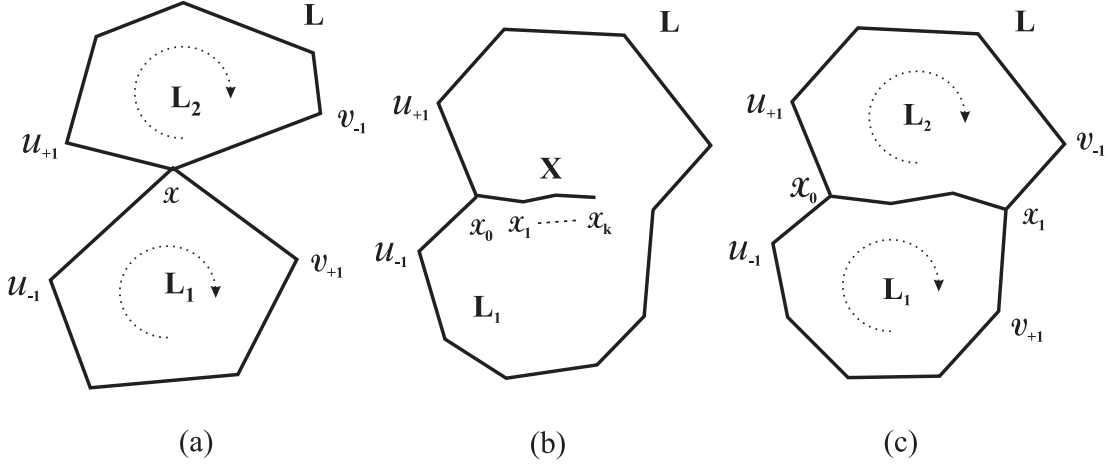


Figure 2: Splitting and pruning of unimodular cycles.

Proposition 11. *Suppose that a cyclic unimodular sequence L has a self-intersection (Figure 2 (a)). If $L = L_1 + L_2$ is the associated shortcut decomposition then $\mu(L) = \mu(L_1) + \mu(L_2)$ and $\nu(L) = \nu(L_1) + \nu(L_2)$.*

Proof: The relation for ν is obvious while the additivity of μ follows from the First Exchange Lemma (Proposition 8). Indeed, the difference $\mu(L) - \mu(L_1) - \mu(L_2)$ reduces to,

$$\mu(u_{-1}, x, u_{+1}) + \mu(v_{-1}, x, v_{+1}) - \mu(u_{-1}, x, v_{+1}) - \mu(v_{-1}, x, u_{+1}) = 0.$$

Proposition 12. *Suppose that $L = \dots u_{-1}x_0x_1 \dots x_k \dots x_1x_0u_{+1} \dots$ is a cyclic unimodular sequence which has a fragment X where it goes back and forth (Figure 2 (b)). Let $L_1 = \dots u_{-1}x_0u_{+1} \dots$ be the cyclic unimodular sequence obtained from L by removing the segment $X = x_0x_1 \dots x_k \dots x_1x_0$. Then,*

$$\mu(L) = \mu(L_1) \quad \text{and} \quad \nu(L) = \nu(L_1).$$

Proof: The second equality is obvious while the first follows from the Jacobi identity for the μ -invariant (Proposition 10). Indeed,

$$\mu(L) - \mu(L_1) = \mu(u_{-1}, x_0, u_{+1}) - \mu(u_{-1}, x_0, x_1) + \mu(x_1, x_0, u_{+1}) = 0$$

by an application of identity (10). \square

3.2 Removing triangles

Suppose that a cyclic unimodular sequence $L = \dots v_{-1}v_0v_{+1} \dots$ has a built-in unimodular triangle $L_1 = v_{-1}v_0v_{+1}$; this situation arises if $\det(v_{-1}, v_{+1}) \in \{-1, +1\}$. Then the sequence L_0 obtained from L by deleting the vector v_0 is also a cyclic unimodular sequence and there is a homological decomposition $L = L_0 + L_1$ (meaning that $[L] = [L_0] + [L_1]$). The following proposition shows that the invariants μ and ν behave in the expected way.

Proposition 13. *Let us assume that a cyclic unimodular sequence L has three consecutive vectors v_{-1}, v_0, v_{+1} such that v_{-1}, v_{+1} is a basis of the lattice \mathbb{Z}^2 . Then L_0 obtained from L by deleting the vector v_0 and the three-term sequence $L_1 = (v_{-1}, v_0, v_{+1})$ are both cyclic unimodular sequences and,*

$$\mu(L) = \mu(L_0) + \mu(L_1) \quad \nu(L) = \nu(L_0) + \nu(L_1) \quad (11)$$

Proof: As before the relation $\nu(L) = \nu(L_0) + \nu(L_1)$ is straightforward. The difference $\mu(L) - \mu(L_0) - \mu(L_1)$ is equal to $A - B - C$ where

$$A = \mu(v_{-2}, v_{-1}, v_0) + \mu(v_{-1}, v_0, v_{+1}) + \mu(v_0, v_{+1}, v_{+2})$$

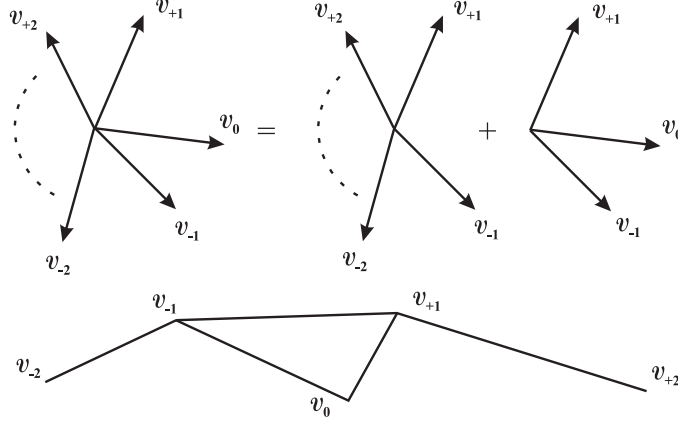


Figure 3: Removing a triangle.

$$B = \mu(v_{-2}, v_{-1}, v_{+1})\mu(v_{-1}, v_{+1}, v_{+2})$$

$$C = \mu(v_{-1}, v_0, v_{+1}) + \mu(v_0, v_{+1}, v_{-1}) + \mu(v_0, v_{+1}, v_{-1}).$$

All terms in $A - B - C$ cancel out as a consequence of the Jacobi identity (Proposition 10, equation (10)) applied on vertices v_{-1} and v_{+1} of the triangle $L_1 = v_1 v_0 v_{+1}$ (Figure 3). \square

3.3 Splitting cycles

Proposition 13 is a special case of a general “unimodular cycle splitting principle” which is also a consequence of the Jacobi identity and the First Exchange Lemma.

Proposition 14. *Suppose that a cyclic unimodular sequence*

$$L = \dots u_{-1} x_0 u_{+1} \dots v_{-1} x_1 v_{+1} \dots$$

(as depicted in Figure 2 (c)) admits a unimodular shortcut $X = x_0 \dots x_1$ giving rise to two new cyclic unimodular sequences $L_1 = \dots u_{-1} X v_{+1} \dots$ and $L_2 = \dots v_{-1} X' u_{+1} \dots$ (where $X' := x_1 \dots x_0$). Then,

$$\mu(L) = \mu(L_1) + \mu(L_2) \quad \text{and} \quad \nu(L) = \nu(L_1) + \nu(L_2).$$

Proof: The second part of the proposition (referring to the invariant ν) is again obvious. The first equality can be proved directly, along the lines of

the proof of Proposition 13, or deduced as a consequence of Propositions 11 and 12.

Indeed, if $X = x_0 Y x_1$ (and $X' = x_1 Y' x_0$) then there exist decompositions $L \cong A x_0 B x_1, L_1 \cong A X, L_2 \cong B X'$, where \cong expresses equality of cyclic unimodular sequences ‘up to a cyclic permutation’. Then by Proposition 12 $\mu(L) = \mu(A x_0 Y x_1 Y' x_0 B)$ and by Proposition 11 $\mu(A x_0 Y x_1 Y' x_0 B) = \mu(L_1) + \mu(L_2)$. \square

3.4 Antipodal unimodular sequences

Antipodal unimodular sequences were in Section 1.1 introduced as the unimodular sequences $U = u_1 u_2 \dots u_{d+1}$ satisfying the condition $u_{d+1} = -u_1$. By associating to such a sequence a polygonal curve we observe that the rotation number $Rot(U)$ is well defined, say with the aid of the formula (2). Moreover, $Rot(U) = (1/2)Rot(U^c)$ where U^c is the centrally symmetric extension $U^c = u_1 \dots u_d (-u_1) \dots (-u_d)$ of U_d .

The invariant μ can be also extended to antipodal unimodular sequences by the formula,

$$\mu(U) = \mu(-u_d, u_1, u_2) + \sum_{i=2}^{d-1} \mu(u_{i-1}, u_i, u_{i+1})$$

or equivalently by $\mu(U) := (1/2)\mu(U^c)$.

Proposition 15. (Antipodality Principle) *All the results from the previous sections, including the Theorem 1, can be extended from the case of cyclic to the case of antipodal unimodular sequences.*

Proof: Apply a result known to hold for cyclic unimodular sequences to the centrally symmetric extension U^c of the antipodal unimodular sequence $U = u_1 u_2 \dots u_{d+1}$ and use the relations

$$\mu(U) := \frac{1}{2}\mu(U^c) \quad \nu(U) = \frac{1}{2} = \nu(U^c) \quad Rot(U) = \frac{1}{2}Rot(U^c).$$

Example 16. As an illustration let us check Theorem 1 for the case of an antipodal unimodular sequence $L = (u, v, -u)$ of length 3. Assume that $\det(u, v) = \det(v, -u) = \epsilon$. Then $Rot(L) = \epsilon/2$, the invariant μ vanishes and $\nu(L) = 2\epsilon$, which is in agreement with (1).

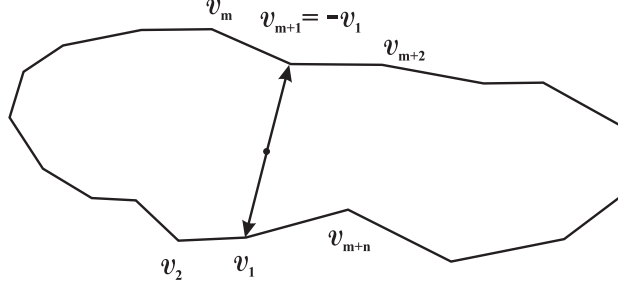


Figure 4: Splitting into antipodal unimodular sequences.

Proposition 17. (Antipodal Reduction Lemma) *Suppose that a cyclic unimodular sequence L has a pair of antipodal elements, for example let $L = v_1 v_2 \dots v_m v_{m+1} \dots v_{m+n}$ where $v_{m+1} = -v_1$. Then $L_1 = v_1 v_2 \dots v_m v_{m+1}$ and $L_2 = v_{m+1} \dots v_{m+n} v_1$ are antipodal unimodular sequences and,*

$$\mu(L) = \mu(L_1) + \mu(L_2) = \frac{1}{2}\mu(L_1^c) + \frac{1}{2}\mu(L_2^c). \quad (12)$$

Proof: The formula is a consequence of the Second Exchange Lemma (Proposition 9). Indeed, the difference $\mu(L) - \mu(L_1) - \mu(L_2)$ reduces to,

$$\mu(v_{m+n}, v_1, v_2) + \mu(v_m, -v_1, v_{m+2}) - \mu(v_{m+n}, v_1, -v_{m+2}) - \mu(v_m, -v_1, -v_2) = 0.$$

4 The proof of Theorem 1

Proof of Theorem 1: We have already observed (Proposition 15) that the equality (1) should be also valid for antipodal unimodular sequences (Section 3.4). We carry on the proof simultaneously for both types of sequences so from here on $L = u_1 u_2 \dots u_d (u_{d+1})$ is either a cyclic unimodular sequence or an antipodal unimodular sequence. In other words L is either closed (the case $u_{d+1} = u_1$) or open in which case it is antipodal ($u_{d+1} = -u_1$).

Definition 18. *A unimodular sequence $L = u_1 u_2 \dots u_d u_{d+1}$ is called proper if $u_i \neq u_j$ and $u_i \neq -u_j$ for each pair $i \neq j$ of indices such that $1 \leq i, j \leq d$.*

We begin with the observation that the proof of the equality

$$Rot(L) = \frac{1}{12}\mu(L) + \frac{1}{4}\nu(L)$$

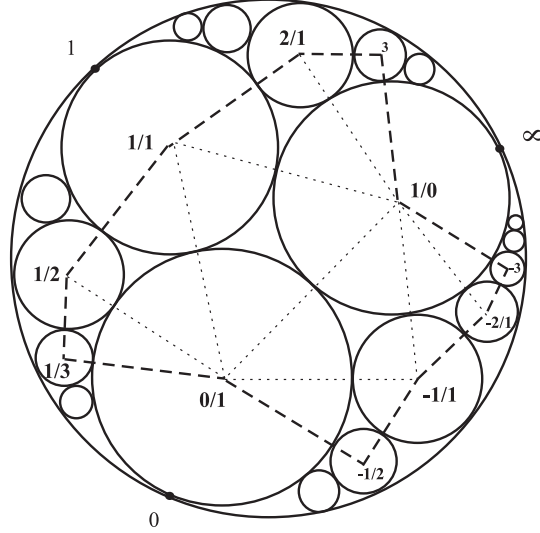


Figure 5: Representing unimodular sequences in the hyperbolic plane.

can be reduced to the case of *proper sequences*.

Indeed, the reduction procedures from previous sections allow us to remove self-intersections (Proposition 11), back-and-forth-segments (Proposition 12), and antipodal vectors different from end-points (Proposition 17). Eventually this procedure produces a finite sequence L_1, \dots, L_k of *proper sequences* such that

$$\text{Rot}(L) = \sum_{i=1}^k \text{Rot}(L_i) \quad \mu(L) = \sum_{i=1}^k \mu(L_i) \quad \nu(L) = \sum_{i=1}^k \nu(L_i). \quad (13)$$

4.1 The case of proper sequences

The map $\phi : \mathbb{Z}^2 \rightarrow \mathbb{Q} \cup \{\infty\}$, associating to a lattice vector $u = (x, y)$ its slope $\phi(u) = x/y$, is a 2-to-1 map on the set $\text{Prim}(\mathbb{Z}^2)$ of primitive vectors. There is a well known picture (Figure 5), related to the action of the modular group $PSL_2(\mathbb{Z})$ in the upper half-plane $H = \{z \in \mathbb{C} \mid \text{Im}(z) \geq 0\}$ (depicted in Figure 5 as a disc), where each rational number p/q is associated a circle (horocycle) $C_{p/q}$ in H . Each primitive lattice vector $u = (p, q)$ is associated a circle $C_u := C_{\phi(u)} = C_{p/q}$ and,

- (1) $C_{u_1} = C_{u_2}$ if and only if $u_1 = \pm u_2$,

- (2) Circles C_{u_1} and C_{u_2} are adjacent if and only if $\{u_1, u_2\}$ is a base of \mathbb{Z}^2 ,
- (3) Circles $C_{u_1}, C_{u_2}, C_{u_3}$ are pairwise adjacent if and only if u_1, u_2, u_3 is a circular unimodular sequence.

Let $L = u_1 u_2 \dots u_d u_{d+1}$ be a proper unimodular sequence which is either cyclic ($u_1 = u_{d+1}$) or antipodal ($u_1 = -u_{d+1}$). If c_u is the center of the circle C_u then $c_{u_1} = c_{u_{d+1}}$ and $c_{u_1}, c_{u_2}, \dots, c_{u_d}, c_{u_{d+1}}$ are vertices of a *simple* closed polygonal curve (Jordan curve) Λ (Figure 5). This curve is a boundary of a polygonal region (a topological disc) which can be triangulated (without new vertices) so that the vertices of each triangle are centers of pairwise adjacent circles. This triangulation is used to construct a splitting $L = L_1 + \dots + L_k$ of the original sequence into cyclic or antipodal sequences of length 3 where the formula (1) is easily verified (Examples 7 and 16). \square

Example 19. The polygonal line depicted in Figure 5 has the sequence

$$c_{0/1}, c_{-1/2}, c_{-1/1}, c_{-2/1}, c_{-3/1}, c_{1/0}, c_{3/1}, c_{2/1}, c_{1/1}, c_{1/2}, c_{1/3}, c_{0/1}$$

as the set of its vertices. This sequence arises as the image of many proper unimodular sequences, one of them is the antipodal unimodular sequence

$$(0, 1), (-1, 2), (-1, 1), (-2, 1), (-3, 1), (-1, 0), \\ (-3, -1), (-2, -1), (-1, -1), (-1, -2), (-1, -3), (0, -1)$$

depicted in Figure 6 (left). By removing the triangles this sequence can be eventually reduced to the central triangle corresponding to the three term unimodular sequence depicted on the right in Figure 6.

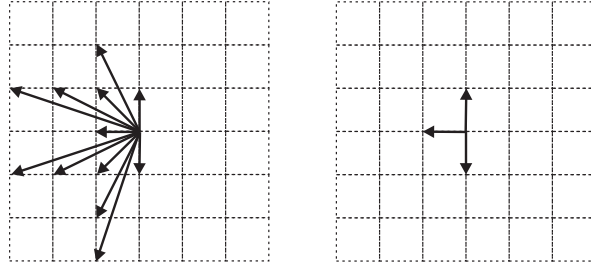


Figure 6: Reduction of a proper antipodal sequence.

5 Appendix

It is quite natural to ask for higher dimensional generalizations and analogues of formula (1).

Problem: Suppose that M^d is an oriented, triangulated, closed manifold. A map $\phi : M^d \rightarrow \mathbb{R}^{d+1}$ is called *unimodular* if for each d -simplex (a_0, a_1, \dots, a_d) of M^d the set $\{\phi(a_0), \phi(a_1), \dots, \phi(a_d)\}$ is a base of the lattice $\mathbb{Z}^{d+1} \subset \mathbb{R}^{d+1}$. If $[M] \in H_d(M^d; \mathbb{Z})$ is the fundamental class of M then

$$\phi_*([M]) \in H_*(\mathbb{R}^{d+1} \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z}$$

and $\phi_*([M]) = k\iota$ for some integer k (ι is the generator of \mathbb{Z}). The problem is to find a formula expressing the integer k in terms of locally defined quantities, analogous to (1).

References

- [C] J. H. Conway, *The sensual (quadratic) form*, Carus Mathematical Monographs, 26, M. A. A. 1997, ISBN 0-88385-030-3, MR 1478672.
- [F] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies 133, Princeton Univ. Press, 1993.
- [H-M] A. Higashitani, M. Masuda, Lattice multi-polygons, arXiv:1204.0088v2 [math.CO], Apr 2012.
- [P-R] B. Poonen, F. Rodrigues-Villegas, Lattice polygons and the number 12, American Mathematical Monthly, Vol. 107, No. 3, pp. 238–250.